Scaling Limit of Vicious Walkers, Schur Function, and Gaussian Random Matrix Ensemble

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We consider the diffusion scaling limit of the vicious walkers and derive the time-dependent spatial-distribution function of walkers. The dependence on initial configurations of walkers is generally described by using the symmetric polynomials called the Schur functions. In the special case in the scaling limit that all walkers are started from the origin, the probability density is simplified and it shows that the positions of walkers on the real axis at time one is identically distributed with the eigenvalues of random matrices in the Gaussian orthogonal ensemble. Since the diffusion scaling limit makes the vicious walkers converge to the nonintersecting Brownian motions in distribution, the present study will provide a new method to analyze intersection problems of Brownian motions in one-dimension.

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The problem of vicious walkers was introduced by Fisher and its application to various wetting and melting phenomena were described in his Boltzmann medal lecture [1]. Recently, by using the standard one-to-one correspondence between walks and Young tableaux, Guttmann *et al.* [2] and Krattenthaler *et al.* [3] showed that the exact solutions for some enumeration problems of vicious walks [4, 5] are derived from the theory of symmetric functions [6] or the representation theory of classical groups [7]. Important analogies between the ensembles of Young tableaux and those of Gaussian random matrices were reported by Johansson [8], and Baik [9] and Nagao and Forrester [10] studied the vicious walker problem using the random matrix theory of the Gaussian orthogonal ensemble (GOE) [11, 12].

The purpose of this Letter is to demonstrate more explicit relations among the vicious walker problem, the Schur function [13], and the GOE, by performing the diffusion scaling limit of the vicious walkers. We derive the time-dependent spatial-distribution function of walkers in this scaling limit. The dependence on the initial configurations is generally described by using the Schur functions. We show that the case, in which all walkers are started from the origin, can be treated, and in this special case the probability density of positions of walkers at time $t=1$ is identified with the probability density of eigenvalues in the GOE. Since the distribution of random walkers converges to that of Brownian motions in the diffusion scaling limit, the present analysis will solve some intersection problems of one-dimensional Brownian motions. More applications to the probability theory will be reported elsewhere [14]

Vicious walks are defined as a subset of the simple random walks as follows. Let $\{R_k^{s_i}\}_{k\geq 0}, i \in I_n \equiv \{1,2,\cdots,n\}$, be the n independent symmetric simple random walks on $\mathbf{Z}=\{\cdots,-2,-1,0,1,2\cdots\}$ started from n distinct positions, $2s_1 < 2s_2 < \cdots < 2s_n$, $s_i \in \mathbf{Z}$. Fix the time interval m as a positive even number. The total number of walks is 2^{mn}, all of which are assumed to be realized with equal probability 2^{-mn}. Now we consider a subset of walks such that any of walkers does not meet other walkers up to time m, that is, the condition

$$R_k^{s_1} < R_k^{s_2} < \cdots < R_k^{s_n} \quad 1 \leq \forall k \leq m \tag{1}$$

is imposed. Such a subset of walks is called the vicious walks (up to time m) [1]. Let $N_n(m;\{e_i\}|\{s_i\})$ be the total number of the vicious walks, in which the n walkers arrive at the positions $2e_1 < 2e_2 < \cdots < 2e_n$ at time m. Then the probability that such vicious walks with fixed end-points are realized in all possible random walks started from the given initial configuration is $N_n(m;\{e_i\}|\{s_i\})/2^{mn}$, which is denoted by $V_n(\{R_k^{s_i}\}_{k=0}^m; R_m^{s_i}=2e_i)$ in this Letter. We also define

$$V_n(\{R_k^{s_i}\}_{k=0}^m) = \sum_{e_1<e_2<\cdots<e_n} V_n(\{R_k^{s_i}\}_{k=0}^m; R_m^{s_i}=2e_i).$$

Recently Krattenthaler *et al.* [3] evaluated the asymptotes for large m of $V_n(\{R_k^{s_i}\}_{k=0}^m)$ for the two special initial-configurations, (i) $s_i = i-1$ and (ii) $s_i = 2(i-1)$, as

$$V_n(\{R_k^{s_i}\}_{k=0}^m) = a_n b_n(\{s_i\}) m^{-n(n-1)/4}\left(1+\mathcal{O}(1/m)\right),$$

where

$$a_n = \begin{cases} (2^n/\pi)^{n/4} \displaystyle\prod_{i=1}^{n/2}(2i-2)! & \text{if } n = \text{even} \\ (2^{n+1}/\pi)^{(n-1)/4} \displaystyle\prod_{i=1}^{(n-1)/2}(2i-1)! & \text{if } n = \text{odd}, \end{cases} \tag{2}$$

with $b_n(\{i-1\}) = 1, b_n(\{2(i-1)\}) = 2^{n(n-1)/2}$.

We found that their result can be immediately generalized as

$$b_n(\{s(i-1)\}) = s^{n(n-1)/2} \text{ for any } s = 1, 2, \cdots. \quad (3)$$

This observation suggests that we can take the scaling limit such that $L \to \infty$ with the time interval $m = Lt$ and the initial spacing of walkers $s = \sqrt{L}/2$, where t is finite.

In this Letter we consider the *diffusion scaling limit*; setting $m = Lt, s_i = \sqrt{L}x_i/2, e_i = \sqrt{L}y_i/2, i \in I_n$, and taking the limit $L \to \infty$. The key lemma, which will be proved shortly, is the following. For given $t > 0$ and $0 \leq x_1 < x_2 < \cdots < x_n$ $(x_i \in \mathbf{Z}), y_1 < y_2 < \cdots < y_n$, let $\xi(x) = (\xi_1(x), \cdots, \xi_n(x))$ be a partition specified by the starting positions $\{x_i\}$ as

$$\xi_i(x) = x_{n-i+1} - (n-i), \ i \in I_n, \quad (4)$$

then

$$\lim_{L\to\infty} \left(\frac{\sqrt{L}}{2}\right)^n V_n \left(\left\{ R_k^{\sqrt{L}x_i/2} \right\}_{k=0}^{Lt}; R_{Lt}^{\sqrt{L}x_i/2} = \frac{\sqrt{L}y_i}{2} \right)$$
$$= (2\pi t)^{-n/2} s_{\xi(x)} \left(e^{y_1/t}, e^{y_2/t}, \cdots, e^{y_n/t} \right)$$
$$\times \exp\left(-\frac{1}{2t} \sum_{i=1}^n (x_i^2 + y_i^2) \right) \prod_{1\leq i<j\leq n} (e^{y_j/t} - e^{y_i/t}), \quad (5)$$

where $s_\lambda(z_1, \cdots, z_n)$ is the Schur function [13]. We consider the rescaled one-dimensional lattice $\mathbf{Z}/(\sqrt{L}/2)$, where the unit length is $2/\sqrt{L}$, and let $\tilde{R}_k^x$ denote the symmetric simple random walk starting from x on $\mathbf{Z}/(\sqrt{L}/2)$. Then (5) implies that

$$\lim_{L\to\infty} V_n \left(\left\{ \tilde{R}_k^{x_i} \right\}_{k=0}^{Lt}; \tilde{R}_{Lt}^{x_i} \in [y_i, y_i + dy_i] \right)$$
$$= f_n(t; \{y_i\}|\{x_i\}) d^n y,$$

where $f_n(t; \{y_i\}|\{x_i\})$ is defined by the RHS of (5).

In order to normalize f_n, we consider the integral

$$\mathcal{N}_n(t; \{x_i\}) = \int_{y_1<\cdots<y_n} d^n y \ f_n(t; \{y_i\}|\{x_i\})$$
$$= \frac{e^{-\sum x_i^2/2t}}{(2\pi t)^{n/2} n!} \int d^n y \ s_{\xi(x)}(e^{y_1/t}, \cdots, e^{y_n/t})$$
$$\times e^{-\sum y_i^2/2t} \prod_{1\leq i<j\leq n} |e^{y_j/t} - e^{y_i/t}|, \quad (6)$$

where we have used the fact that with the absolute values the integrand is invariant under permutation of y_i, since the Schur function is a symmetric function. The spatial-distribution function in the scaling limit of the n vicious walkers at finite time t is then given by $d\mu_n = f_n d^n y/\mathcal{N}_n$, or more explicitly

$$d\mu_n(t; \{y_i\}|\{x_i\}) = g_n(t; \{y_i\}|\{x_i\}) d^n y,$$

with the probability density

$$g_n(t; \{y_i\}|\{x_i\}) = \frac{\mathbf{1}\{y_1 < y_2 < \cdots < y_n\}}{Z_n(t; \{x_i\})}$$
$$\times s_{\xi(x)}(e^{y_1/t}, \cdots, e^{y_n/t})$$
$$\times \exp\left(-\frac{1}{2t} \sum_{i=1}^n y_i^2 \right) \prod_{1\leq i<j\leq n} (e^{y_j/t} - e^{y_i/t}) \quad (7)$$

for $0 \leq x_1 < \cdots < x_n, x_i \in \mathbf{Z}$, where $\mathbf{1}\{\omega\}$ is 1 if ω is satisfied and is zero otherwise, and $Z_n(t; \{x_i\}) = (2\pi t)^{n/2} e^{\sum x_i^2/2t} \mathcal{N}_n(t; \{x_i\})$. It should be noted that, if the initial configuration is $x_i = i-1, i \in I_n$, then $\xi_i(x) \equiv 0$ and the Schur function in (6) and (7) is $s_{\xi(x)} = 1$.

Now we give a proof of (5). Define a subset of the square lattice $\mathbf{Z}^2$,

$$\mathcal{L}_m = \{(x, y) \in \mathbf{Z}^2 : x + y = \text{even}, \ 0 \leq y \leq m\},$$

and $\mathcal{E}_m$ be the set of all edges which connect the nearest-neighbor pairs of vertices in $\mathcal{L}_m$. Then each walk of the i-th walker, $i \in I_n$, can be represented as a sequence of successive edges connecting vertices $S_i = (2s_i, 0)$ and $E_i = (2e_i, m)$ on $(\mathcal{L}_m, \mathcal{E}_m)$, which we call the *lattice path* running from S_i to E_i. If such lattice paths share a common vertex, they are said to intersect. Under the vicious walk condition (1), what we consider is a set of all n-tuples of *nonintersecting paths* [15]. Let $\pi(S \to E)$ be the set of all lattice paths from S to E, and $\pi_0(\{S_i\}_{i=1}^n \to \{E_i\}_{i=1}^n)$ be the set of all n-tuples $(\pi_1, \cdots, \pi_n)$ of nonintersecting lattice paths, in which π_i runs from S_i to E_i, $i \in I_n$. If we write the number of elements in a set A as $|A|$, then $N_n(m; \{e_i\}|\{s_i\}) = |\pi_0(\{S_i\}_{i=1}^n \to \{E_i\}_{i=1}^n)|$ and the Lindström-Gessel-Viennot theorem gives [15, 16],

$$N_n(m; \{e_i\}|\{s_i\}) = \det_{1\leq i,j\leq n}(|\pi(S_j \to E_i)|).$$

Since $|\pi(S_j \to E_i)| = \binom{m}{m/2+s_j-e_i}$, we have the binomial determinant

$$V_n(\{R_k^{s_i}\}_{k=0}^m; R_m^{s_i} = 2e_i)$$
$$= 2^{-mn} \det_{1\leq i,j\leq n} \left(\binom{m}{m/2 + s_j - e_i} \right).$$

Application of Stirling's formula yields

$$\lim_{L\to\infty} 2^{-Ltn} (\sqrt{L}/2)^n \det_{1\leq i,j\leq n} \left(\binom{Lt}{Lt/2 + \sqrt{L}(x_j - y_i)/2} \right)$$
$$= \det_{1\leq i,j\leq n} \left(\lim_{L\to\infty} 2^{-Lt} (\sqrt{L}/2) \binom{Lt}{Lt/2 + \sqrt{L}(x_j - y_i)/2} \right)$$
$$= \det_{1\leq i,j\leq n} \left((2\pi t)^{-1/2} \ e^{-(x_j - y_i)^2/2t} \right)$$
$$= (2\pi t)^{-n/2} e^{-\sum (x_i^2 + y_i^2)/2t} \det_{1\leq i,j\leq n} \left(e^{x_j y_i/t} \right).$$

We rewrite the determinant as

$$\det_{1\le i,j\le n}\left(e^{x_j y_i/t}\right) = \frac{\det_{1\le i,j\le n}\left((e^{y_i/t})^{x_{n-j+1}}\right)}{\det_{1\le i,j\le n}\left((e^{y_i/t})^{n-j}\right)}\times \Delta_n\left(\{e^{y_i/t}\}\right),$$

where $\Delta_n(\{z_i\})$ is the Vandermonde determinant

$$\Delta_n(\{z_i\}) \equiv \det_{1\le i,j\le n}(z_i^{j-1}) = \prod_{1\le i<j\le n}(z_j - z_i).$$

Combining with (4) and the definition of Schur function [13] completes the proof of (5).

Since

$$s_{\xi(\ell x)}(e^{y_1/\ell t},\cdots,e^{y_n/\ell t})\Delta(\{e^{y_i/\ell t}\}) = s_{\xi}(x)(e^{y_1/t},\cdots,e^{y_n/t})\Delta(\{e^{y_i}\}) = \det_{1\le i,j\le n}\left(e^{x_j y_i/t}\right)$$

for any integer ℓ, where $\ell x=(\ell x_1,\cdots,\ell x_n)$, we can prove the following *scaling property* for the scaling-limit probability density. For any integer ℓ

$$g_n(t;\{y_i\}|\{x_i\}) = \ell^n g_n(\ell^2 t;\{\ell y_i\}|\{\ell x_i\}). \tag{8}$$

Using this property, we can generalize the expression (7) for any rational numbers, $x_1<\cdots<x_n$, and then using the connectedness of real numbers, for any real numbers $\{x_i\}$.

Next we study the $t\to\infty$ asymptotes of the above results. Since [13]

$$\lim_{t\to\infty} s_{\xi(x)}(e^{y_1/t},\cdots,e^{y_n/t}) = s_{\xi(x)}(1,1,\cdots,1) = \prod_{1\le i<j\le n}\frac{x_j-x_i}{j-i}\equiv b_n(\{x_i\})$$

(remark that this definition of $b_n(\{x\})$ is consistent with (3)), and

$$\lim_{t\to\infty} t^{n(n-1)/2}\prod_{1\le i<j\le n}(e^{y_j/t}-e^{y_i/t}) = \prod_{1\le i<j\le n}(y_j-y_i),$$

the normalization factor is asymptotically

$$\begin{aligned}\mathcal{N}_n(t;\{x_i\}) &= t^{-n^2/2}\left(1+\mathcal{O}(1/t)\right)\\ &\quad\times\frac{b_n(\{x_i\})}{(2\pi)^{n/2}n!}\int d^n y\, e^{-\sum y_i^2/2t}\prod_{1\le i<j\le n}|y_j-y_i|\\ &= t^{-n(n-1)/4}\left(1+\mathcal{O}(1/t)\right)\\ &\quad\times\frac{b_n(\{x_i\})}{(2\pi)^{n/2}n!}\int d^n u\, e^{-\sum u_i^2/2}\prod_{1\le i<j\le n}|u_j-u_i|,\end{aligned}$$

where $u_i = y_i/\sqrt{t}$. The last integral is the special case ($\gamma=1/2$ and $a=1/2$) of

$$\int d^n u\, e^{-a\sum u_i^2}\prod_{1\le i<j\le n}|u_j-u_i|^{2\gamma} = (2\pi)^{n/2}(2a)^{-n(\gamma(n-1)+1)/2}\prod_{i=1}^{n}\frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)},$$

which was derived by Mehta (eq.(17.6.7) on page 354 in [11]) as a consequence of Selberg's integral. Here $\Gamma(x)$ is the Gamma function with the values $\Gamma(3/2)=\sqrt{\pi}/2$ and $\prod_{i=1}^n\Gamma(1+i/2) = 2^{-n(n-1)/2}(\sqrt{\pi}/2)^n n!\; a_n$, where a_n is given by (2). Then we have

$$\begin{aligned}&\mathcal{N}_n(t;\{x_i\})\\ &= t^{-n(n-1)/4}2^{-n(n-1)/2}a_n b_n(\{x_i\})\left(1+\mathcal{O}(1/t)\right)\end{aligned} \tag{9}$$

as t tends to infinity. Using (9) the asymptotic form in $t\to\infty$ of the probability density is given as

$$\begin{aligned}g_n(t;\{y_i\}|\{x_i\}) &= \mathbf{1}\{y_1<y_2<\cdots<y_n\}c_n t^{-n(n+1)/4}\\ &\quad\times e^{-\sum y_i^2/2t}\prod_{1\le i<j\le n}(y_j-y_i)\ \left(1+\mathcal{O}(1/t)\right)\end{aligned}$$

with $c_n = 2^{n(n-1)/2}/\{(2\pi)^{n/2}a_n\}$ and thus

$$\begin{aligned}&\lim_{t\to\infty} d\mu_n(t;\{\sqrt{t/\tau}u_i\}|\{x_i\}) = \mathbf{1}\{u_1<u_2<\cdots<u_n\}\\ &\quad\times c_n\tau^{-n(n+1)/4}\ e^{-\sum u_i^2/2\tau}\prod_{1\le i<j\le n}(u_j-u_i)d^n u\end{aligned} \tag{10}$$

for any $\tau>0$. We note that the diffusion scaling limit, $L\to\infty$ with $m=Lt, s_i=\sqrt{L}x_i/2, e_i=\sqrt{L}y_i/2$, and the above $t\to\infty$ limit with $y_i=\sqrt{t/\tau}u_i$ are combined to define a limit $L'\equiv Lt/\tau\to\infty$ with $m=L'\tau, e_i=\sqrt{L'}u_i/2$ and s_i in the smaller order than $\sqrt{L'}$. Since $s_i/e_i\to 0$ in this limit, the limit (10) should be regarded as the scaling limit $d\mu_n(t;\{y_i\}|\{x_i\})$ for $x_i=0, i\in I_n$. That is, when all walkers are started from the origin, the probability density in the scaling limit should be

$$\begin{aligned}g_n(t;\{y_i\}|\{0\}) &= \mathbf{1}\{y_1<y_2<\cdots<y_n\}\ c_n t^{-n(n+1)/4}\\ &\quad\times\exp\left(-\frac{1}{2t}\sum_{i=1}^n y_i^2\right)\prod_{1\le i<j\le n}(y_j-y_i).\end{aligned} \tag{11}$$

By using the scaling property (8), we can give another explanation for the reason why the above limit procedure gives the case for $x_i\equiv 0$ as follows. The scaling property (8) can be written for distribution functions as

$$d\mu_n(t;\{y_i\}|\{x_i\}) = d\mu_n(\ell^2 t;\{\ell y_i\}|\{\ell x_i\})$$

for any integer ℓ. Then, if we set $x_i\equiv 0$ and $T=\ell^2 t$, we have the invariance

$$d\mu_n(t;\{y_i\}|\{0\}) = d\mu_n(T;\{\sqrt{T/t}\ y_i\}|\{0\}) \tag{12}$$

for any $T>0$. Then we can take the limit $T\to\infty$, which will derive the expression (11).

The result (11) is very important, since if we set $t=1$ and assume that $y_1<\cdots<y_n$, then we have the equality

$$g_n(1;\{y_i\}|\{0\}) = n!\ g_n^{\mathrm{GOE}}(\{y_i\})$$

where $g_n^{\mathrm{GOE}}(\{y_i\})$ is the probability density of eigenvalues of random matrices in the GOE [11]. The equality

(12) with $T=1$ implies that $d\mu_n(t;\{y_i\}|\{0\})$ is a function of n variables $\{u_i = y_i/\sqrt{t}\}$. In other words, the dynamical scaling with the classical exponent $\theta=2$ (see, for example, chapter 16 in [17]) is established, in which the scaling function is exactly and explicitly given by $n! \times g_n^{\rm GOE}(\{u_i\})$ with the scaling variables $u_i = y_i/t^{1/\theta}$.

In the present diffusion scaling limit, the random walks converge to the Brownian motions in distribution. Then (9) gives the asymptote in $t \to \infty$ of the probability that the n independent one-dimensional Brownian motions started from $x_1 < x_2 < \cdots < x_n$ ($x_i \in \mathbf{Z}$) do not intersect up to time t. The scaling property (8) is nothing but the diffusion scaling of Brownian motions, $\mathcal{N}_n(\ell^2 t; \{\ell x_i\}) = \mathcal{N}_n(t;\{x_i\})$. Moreover, we can define a diffusion process by giving the transition probability density as

$$\begin{aligned} p_n(s;\{y_i\}|\{x_i\}) &= \lim_{t\to\infty} \frac{f_n(s;\{y_i\}|\{x_i\})\mathcal{N}_n(t;\{y_i\})}{\mathcal{N}_n(t+s;\{x_i\})} \\ &= \frac{b_n(\{y_i\})}{b_n(\{x_i\})} f_n(s;\{y_i\}|\{x_i\}), \end{aligned}$$

where (9) has been used [14]. Isozaki and Yoshida [18] studied the $n=2$ case as the limit process of the two friendly walkers in the “wetting phase”.

In summary, we have performed the diffusion scaling limit of the vicious walkers and determined the time-dependent spatial-distribution functions in the limit. There the Schur function plays an important role to describe the initial-configuration dependence. The relation with the Gaussian orthogonal ensemble of random matrices was clarified. In the present study we have constructed new continuous-time processes on the real axis for arbitrary numbers of walkers n. We remark that the results reported in this Letter may hold also in the fully “wetting phases” [19] of the models of friendly walkers [20, 21, 22, 23], which are related with a famous unsolved problem, the directed percolation problem. Further study of the distribution functions and diffusion processes derived in this Letter will be desired.

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[13] A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a non-increasing series of non-negative integers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The Schur function $s_\lambda(z_1, \cdots, z_n)$ specified by λ is a polynomial of $(z_1, \cdots, z_n) \in V \equiv$ the n dimensional complex vector space, defined by $s_\lambda(z_1,\cdots,z_n) = \det_{1\leq i,j\leq n}\left(z_i^{\lambda_j+n-j}\right) / \det_{1\leq i,j\leq n}\left(z_i^{n-j}\right)$. It is a symmetric polynomial of degree $\sum_{i=1}^n \lambda_i$ in $z_1, \cdots, z_n$ and it is know that (see p.44 in [6]) $s_\lambda(1,1,\cdots,1) = \prod_{1\leq i<j\leq n}(\lambda_i - \lambda_j + j - i)/(j-i)$. The Schur function is a character of the irreducible representation specified by λ of the group $GL(V)$ and its value at $z_1 = \cdots = z_n = 1$ is the dimension of the representation (see, for example, Lecture 6 in [7]).

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